

The Dirac field and the possible origin of gravity.

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The spin connections of the Dirac field have three ingredients that are connected with the Ricci rotations, the Maxwell field, and an *axial field* which minimally interacts with the axial current. I demonstrate that the axial field provides an effective mechanism of auto-localization of the Dirac field into compact objects. The condition that these objects are stable (the energy-momentum is self-adjoint) leads to Einstein's field equations. The Dirac field with its spin connection seem to be a natural material carrier of the space-time continuum in which compact objects are moving along geodesic lines. The long-distance effect of the axial field is indistinguishable from Newton's gravity, which reveals the microscopic nature of gravity and the origin of the gravitational mass.

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1. In this letter, I study the conditions under which the Dirac field can form compact objects, i.e. particles that move along world lines. The first key element of this study is the notion of parallel transport of the Dirac field. All physical properties of the spinor field are encoded in its tensor observables, some of which are considered as internal. They should be transported with the particle and cannot change if the particle is moving in free space. The second key element is the requirement that the observables must be represented by self-adjoint operators.

It is useful to note that the absolute differential, $DV_a \equiv D_c V_a ds^c$, of a vector V_a is the principal linear part of the vector increment with respect to its change in the course of a parallel transport along the same infinitesimal path; parallel transport just means that $DV_a = 0$. The absolute differential of the Dirac field is needed for exactly the same reason. A stable spinor object does not change when it is parallel-transported along its world line.

When dealing with spinors, one has to use the tetrad formalism [1], and the spinors should always be treated as coordinate scalars. The Dirac matrices are pure number constructs (I use them in the original form that was suggested by Dirac [2]). The affine connections for vectors can be derived in a relatively simple way because the rotation of a vector at a given point follows the rotation of the local coordinate axes. There is no similar rule for spinors since their components are not tensor variables. Addressing this problem, I closely follow Fock's method [3] and require the probability current, $j_a = \psi^\dagger \alpha_a \psi = \psi^\dagger (1, \rho_3 \sigma_i) \psi$, to be a Lorentz vector. This vector must be transformed as $j_a(x) \rightarrow \Lambda_a^b(x) j_b(x)$ under a local Lorentz rotation, and the variation of its tetrad components under a parallel displacement ds^a should be $\delta j_a = \omega_{acb} j^c ds^b$, where $\omega_{bca} = -\omega_{cba}$ are the Ricci rotation coefficients fixed by the fact that the tetrad vectors are covariantly constant, i.e., $D_\mu e_\nu^b = \nabla_\mu e_\nu^b - e_\mu^a \omega_{ca}^b e_\nu^c = 0$, and $\nabla_\lambda e_\mu = \partial_\lambda e_\mu - \Gamma_{\mu\lambda}^\nu e_\nu$.

Let the matrix Γ_a (the spin connection) define the change of the spinor components in the course of the same infinitesimal displacement, $\delta\psi = \Gamma_a \psi ds^a$, $\delta\psi^\dagger =$

$\psi^\dagger \Gamma_a^\dagger ds^a$. This gives yet another expression for δj_a , $\delta j_a = \psi^\dagger (\Gamma_b^\dagger \alpha_a + \alpha_a \Gamma_b) \psi ds^b$. The two forms of δj_a must be the same. Hence, the equation that defines Γ_a is

$$\Gamma_b^\dagger \alpha_a + \alpha_a \Gamma_b = \omega_{acb} \alpha^c, \quad (1)$$

and its most general solution is,

$$\Gamma_b(x) = -ieA_b(x) - ig\rho_3 \aleph_b(x) + \Omega_b(x), \quad (2)$$

where $A_a(x)$ and $\aleph_a(x)$ are real vector fields, and $\Omega_b(x) = (1/4)\omega_{cdb}(x)\rho_1\alpha^c\rho_1\alpha^d$ is the geometric part. The covariant derivative of a spinor now reads as $D_a\psi = (\partial_a - \Gamma_a)\psi$ and $D_\mu\psi = e_\mu^a D_a\psi = (\partial_\mu - \Gamma_\mu)\psi$, in the tetrad and co-ordinate basis, respectively.

The absolute differential of any observable spinor form O which is defined by an operator \mathcal{O} is

$$DO(x) \equiv D[\psi^\dagger \mathcal{O} \psi] = \psi^\dagger (\mathcal{O} \overrightarrow{D}_a + \overleftarrow{D}_a^\dagger \mathcal{O}) \psi ds^a. \quad (3)$$

I extensively use this rule throughout this work because it defines a stable and, ultimately, compact spinor object. The parallel transport of O just means that $DO = 0$.

If $\mathcal{J}_a = \psi^\dagger \rho_3 \alpha_a \psi$ is the axial current, and $\mathcal{S} = \psi^\dagger \rho_1 \psi$ and $\mathcal{P} = \psi^\dagger \rho_2 \psi$ are two Lorentz scalars then

$$D_\mu j^\nu = \nabla_\mu j^\nu, \quad D_\mu \mathcal{J}^\nu = \nabla_\mu \mathcal{J}^\nu, \\ D_\mu \mathcal{S} = \nabla_\mu \mathcal{S} + 2g\aleph_\mu \mathcal{P}, \quad D_\mu \mathcal{P} = \nabla_\mu \mathcal{P} - 2g\aleph_\mu \mathcal{S}. \quad (4)$$

The first of these equations just duplicates the input for Eq. (1). The parallel transport of vector currents that are built with the aid of the diagonal Dirac matrices σ_i and ρ_3 is not affected by the axial field, while for the Lorentz scalars, the left and right spinors are mixed by either ρ_1 or ρ_2 , which makes their covariant derivative dependent on \aleph_μ .

2. Taking the linear relation, $u_\mu P^\mu = m$, as a classical prototype for the equation of motion, we can write the following version of the Dirac equation,

$$\alpha^a (\partial_a \psi - \Gamma_a \psi) + im\rho_1 \psi = 0. \quad (5)$$

This equation includes an additional *axial* vector field $\aleph_a(x)$, which is minimally coupled to the axial current and acts differently on different spinor components. The differential operator of the Dirac equation is Hermitian (symmetric), which is confirmed by the conservation of the probability current j^μ . However, since in the most general form of the spin connection the field \aleph can be singular, it is not necessarily a self-adjoint operator. The Lorentz invariant condition that iD_μ is a self-adjoint operator is as follows,

$$\begin{aligned} i \int \partial_\sigma (\sqrt{-g} \psi^\dagger [\alpha^\sigma \overrightarrow{D}_\mu + \overleftarrow{D}_\mu^\dagger \alpha^\sigma] \psi) d^3 \vec{x} dx^0 \\ = i \int \partial_\sigma \nabla_\mu (\sqrt{-g} \psi^\dagger \alpha^\sigma \psi) d^3 \vec{x} dx^0 = 0. \end{aligned} \quad (6)$$

3. The following identities directly result from the equation of motion (5) and its conjugate,

$$\nabla_\mu j^\mu = 0, \text{ and } \nabla_\mu \mathcal{J}^\mu = 2m\mathcal{P}. \quad (7)$$

The first of these identities is the conservation of the time-like probability current. It provides the definition of a scalar product in the space of the Dirac spinors as an integral over the three-dimensional space-like surface. By virtue of the first of Eqs. (4) it also means that the probability density *is* parallel-transported along the vector of probability current, $D_\mu j^\mu = 0$. The second identity tells us that the axial current cannot be conserved (indeed, $\mathcal{J}^2 = -j^2 < 0$; this vector is space-like).

Let us now introduce a standard energy-momentum tensor which is the flux of the 4-momentum density,

$$T^\sigma_\mu = i\psi^\dagger \alpha^\sigma \overrightarrow{D}_\mu \psi. \quad (8)$$

Its operator must also be self-adjoint so that the solutions of the Dirac equation and its adjoint belong to the same space (e.g., have the same spectra of energies). This condition has to be parallel-transported with the compact object that “owns” this spectrum, and the covariant form of this requirement is

$$D_\sigma \{ \psi^\dagger [\alpha^\sigma \overrightarrow{D}_\mu + \overleftarrow{D}_\mu^\dagger \alpha^\sigma] \psi \} = 0, \quad (9)$$

which can be identically transformed into

$$\partial_\sigma (\sqrt{-g} [\psi^\dagger \alpha^\sigma \overrightarrow{D}_\mu \psi + \psi^\dagger \overleftarrow{D}_\mu^\dagger \alpha^\sigma \psi]) + R_{\mu\sigma} \sqrt{-g} j^\sigma = 0, \quad (10)$$

where $R_{ad} = g^{bc} R_{ab;cd}$ is the Ricci curvature tensor. Now, we integrate (10) over the space-time domain, and compare the result with the condition (6). This comparison shows that the Dirac field has a self-adjoint Hamiltonian only if it lives in *free space*. Since $j^\mu \neq 0$, we must have

$$R_{\lambda\sigma} = 0. \quad (11)$$

In general relativity theory, this is Einstein’s equation in “empty” space.

It is straightforward to show (e.g., using the technique of Ref. [3]) that, by virtue of the equations of motion, the following identity holds

$$\nabla_\sigma T^\sigma_\mu = -e F_{\sigma\mu} j^\sigma - g \mathcal{U}_{\sigma\mu} \mathcal{J}^\sigma - 2gm \aleph_\mu \mathcal{P} + \frac{i R_{\mu\sigma}}{2} j^\sigma, \quad (12)$$

where the commutator of the covariant derivatives is expressed in terms of two field strength tensors, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\mathcal{U}_{\mu\nu} = \partial_\mu \aleph_\nu - \partial_\nu \aleph_\mu$, and the Ricci curvature tensor, R_{ab} . By virtue of (11) the last term in (12) is zero, which renders the energy-momentum of the Dirac field a real function.

4. The equations for the fields A_a and \aleph_a that comply with the data (hydrogen spectra) and with the position of these fields in the spin connection are as follows:

$$\nabla_\sigma F^{\sigma\mu} = e j^\mu, \quad (13)$$

so that $F_{\mu\nu}$ is a massless gradient-invariant Maxwell field which has the *probability current* as its source. For the axial field \aleph_μ a plausible choice is a massive neutral vector field,

$$\nabla_\sigma \mathcal{U}^{\sigma\mu} + M^2 \aleph^\mu = g \mathcal{J}^\mu, \quad (14)$$

which sufficiently describes the parity nonconservation phenomena in atoms [4]. The Lorentz forces in Eq. (12), with the field tensors $F_{\mu\nu}$ and $\mathcal{U}_{\mu\nu}$ in a familiar position, prompt the same equations of motion, because these equations allow one to present the Lorentz force as the divergence of the energy-momentum tensor. We have

$$e j^\sigma F_{\sigma\mu} = \nabla_\lambda [F^{\lambda\nu} F_{\nu\mu} + \frac{1}{4} \delta_\mu^\lambda F^{\rho\nu} F_{\rho\nu}] = \nabla_\lambda \Theta^\lambda_\mu. \quad (15)$$

Using (14) and (7) one can reduce the Lorentz force of the axial field in (12) to

$$g \mathcal{J}^\sigma \mathcal{U}_{\sigma\mu} + g \aleph_\mu (\nabla_\sigma \mathcal{J}^\sigma) = \nabla_\lambda (\theta^\lambda_\mu + t^\lambda_\mu), \quad (16)$$

where

$$\theta^\lambda_\mu = \mathcal{U}^{\lambda\nu} \mathcal{U}_{\nu\mu} + \frac{\delta_\mu^\lambda}{4} \mathcal{U}^{\rho\nu} \mathcal{U}_{\rho\nu}, \quad t^\lambda_\mu = M^2 (\aleph^\lambda \aleph_\mu - \frac{\delta_\mu^\lambda}{2} \aleph^\rho \aleph_\rho). \quad (17)$$

In both (13) and (14) we followed a standard convention that the charge is a divergence of electric field. Then the positive charge corresponds to the positive flux of the electric field outside a surrounding surface. With this convention, the energy components Θ^{00} , θ^{00} and t^{00} of tensors (15) and (17) come out *positive* exclusively because the coupling constants in the electron spin connection (2) are chosen *negative*. Assembling Eqs. (12)-(17), one finds that

$$\nabla_\lambda (T^\lambda_\mu + \Theta^\lambda_\mu + \theta^\lambda_\mu + t^\lambda_\mu) = 0, \quad (18)$$

i.e., the total energy-momentum of the interacting fields ψ , A_μ , and \aleph_μ is conserved, which is an additional indication that the system of equations of motion is self-consistent.

5. Special relativity is built on two premises, i.e., the light-like propagation of all fields that carry signals and the existence of inertial frames. The first one is readily implemented by a spinor representation of the Lorentz group: the two-component Weyl spinors just map the light cone since their current is light-like. The second one is more difficult to implement because the solutions of the relativistic wave equations are not easily localized to the extent where they can serve as the observers (rods and clocks) of special relativity. At the same time, the data indicate that the truly localized interactions are due to the spinor fields.

An example of a relativistic point-like particle with “built in” polarization $\vec{\zeta}$ [5] is helpful. If $\vec{\zeta}$ of the rest frame is a part of a 4-vector which is orthogonal to the 4-velocity, $\zeta^\mu u_\mu = 0$, then both vectors are converted into vector fields *by means of* their parallel transport,

$$Du^\mu = (\partial_\nu u^\mu + \Gamma_{\lambda\nu}^\mu u^\lambda) u^\nu d\tau = 0,$$

$$D\zeta^\mu = (\partial_\nu \zeta^\mu + \Gamma_{\lambda\nu}^\mu \zeta^\lambda) u^\nu d\tau = 0.$$

These are the equations for the geodesic trajectory of a point-like particle and for the parallel transport of its polarization along this trajectory, respectively. The vector $\vec{\zeta}$ is a prototype for the *internal polarization* of the Dirac spinor fields which is represented by various bilinear forms, like \mathcal{S} , \mathcal{P} , etc. As long as we wish to treat the electron as a compact object with an internal polarization (quantum numbers including spin, charge, etc.) we have to impose the *requirement* of parallel transport,

$$DS = 0, \quad D\mathcal{P} = 0, \quad \dots \quad (19)$$

for these quantities. It is imposed on the forms in which the left- and right- spinors are mixed by the matrices ρ_1 or ρ_2 . The Ansatz (19) serves as an additional condition on the spin connection $\rho_3 \aleph_a(x)$ of the Dirac field, which makes \aleph_a compatible with the existence of a freely moving stable spinor particle. With an *a priori* form (2) of the spin connection, (19) can be cast into the following coordinate form,

$$D_\mu \mathcal{S} = \nabla_\mu \mathcal{S} + 2g \aleph_\mu \mathcal{P} = 0, \quad D_\mu \mathcal{P} = \nabla_\mu \mathcal{P} - 2g \aleph_\mu \mathcal{S} = 0. \quad (20)$$

Now, instead of being an arbitrary external field that acts on a Dirac spinor, the field \aleph_μ becomes a functional of spinor forms. In order to find this functional note that from Eqs. (20) it follows that $\mathcal{S} \partial_\mu \mathcal{S} + \mathcal{P} \partial_\mu \mathcal{P} = 0$. Hence, the scalar function $\mathcal{R}^2 = \mathcal{S}^2 + \mathcal{P}^2$, is the first integral of these two equations. Since \mathcal{S} and \mathcal{P} are real functions, we may look for solutions in the forms of $\mathcal{S} = \mathcal{R} \cos \Upsilon$, and $\mathcal{P} = -\mathcal{R} \sin \Upsilon$, which yields the following equation,

$$2g \aleph_\mu = -\partial_\mu \Upsilon. \quad (21)$$

Because $\mathcal{R}^2 = j^a j_a$ is the squared probability current, a positive \mathcal{R} is a natural measure of the *localized* spinor

matter. *The potential function $\Upsilon(x)$ remains a distinct characteristic of the Dirac field even in that part of space where $\mathcal{R} \rightarrow 0$.* For perfectly stable spinor matter, the axial field \aleph_μ is completely defined by one scalar function. Its curvature tensor vanishes, $\mathcal{U}^{\sigma\mu} = 0$. Furthermore, if the conditions (19) are exact, then the Eqs. (17) become $\theta_\mu^\lambda = 0$, and

$$t_{\lambda\mu} = (M^2/8g^2)(2 \partial_\lambda \Upsilon \partial_\mu \Upsilon - g_{\lambda\mu} \partial_\rho \Upsilon \partial^\rho \Upsilon). \quad (22)$$

The equation for the field $\Upsilon(x)$ can be found by computing the covariant divergence of Eq. (21) and expressing the divergence of the axial field via the pseudoscalar density, $M^2 \nabla_\mu \aleph^\mu = 2gm\mathcal{P} = -2gm\mathcal{R} \sin \Upsilon$, i.e.,

$$\square \Upsilon(x) - (4g^2 m/M^2) \mathcal{R}(x) \sin \Upsilon(x) = 0. \quad (23)$$

Within a domain where $\mathcal{R}(x)$ is constant, this is the well-known sine-Gordon equation which has soliton-type solutions. This equation is classical, the Planck constant has cancelled out; therefore, the field $\Upsilon(x)$ does not vanish in the classical limit. In the case of a variable source \mathcal{P} , the variations of $\Upsilon(x)$ propagate according to the d'Alembert equation. The Dirac equation itself becomes

$$\alpha^a \{ \partial_a + ieA_a + \frac{i}{2} \rho_3 \partial_a \Upsilon - \Omega_a \} \psi + im\rho_1 \psi = 0, \quad (24)$$

and we have a *nonlinear system* of Eqs.(23)-(24) in which the scale parameter m can no longer be arbitrary. Furthermore, the Dirac equation is singular. Indeed, let Eq.(23) has a spherically symmetric static solution. If the exterior of the domain $r < r_{max}$ is an empty space where $\mathcal{R}^2 = j_0^2 = 0$, then the function Υ is a solution of the external problem for the Laplace equation and when $r > r_{max}$ we have

$$\Upsilon(r) = -\frac{1}{r} \frac{4g^2 m}{M^2} \int_0^{r_{max}} \mathcal{P}(r) r^2 dr = -\frac{\mathcal{Q}}{r}, \quad (\mathcal{Q} \gtrless 0). \quad (25)$$

The potential, which is as singular as $-\partial_r \Upsilon \propto -\mathcal{Q}/r^2$, in the Dirac equation brings about the advent of the “falling onto the centre” phenomenon. This leads to a super-critical binding which initiates a spectrum of (quasi)bound states where the Dirac field can have negative energy.

6. Following the same logic, we have to require that a compact spinor object localizes its energy and momentum along its world line and to add a new element,

$$D_\sigma T_\mu^\sigma = 0, \quad (26)$$

to the Ansatz (19). The kinetic 4-momentum is parallel-transported with the particle when it is moving in free space. Then, by definition,

$$D_\sigma T_\mu^\sigma \equiv i (\psi^\dagger \overleftarrow{D}_\sigma^\dagger \alpha^\sigma \overrightarrow{D}_\mu \psi + \psi^\dagger \alpha^\sigma \overleftarrow{D}_\mu \overrightarrow{D}_\sigma \psi) = 0, \quad (27)$$

which can be identically transformed to

$$\partial_\sigma (\sqrt{-g} T_\mu^\sigma) = i \sqrt{-g} \psi^\dagger \alpha^\sigma [D_\mu D_\sigma - D_\sigma D_\mu] \psi, \quad (28)$$

and compared with the identity (12) that follows from the equations of motion,

$$(-g)^{-1/2} \partial_\sigma [\sqrt{-g} T^\sigma_\mu] = \Gamma^\sigma_{\mu\nu} T^\nu_\sigma - 2mg \aleph_\mu \mathcal{P} + i\psi^+ [D_\sigma D_\mu - D_\mu D_\sigma] \psi.$$

These two equations coincide if the force of inertia and the external force from the axial field \aleph are equal,

$$\Gamma^\sigma_{\mu\nu} T^\nu_\sigma = 2mg \mathcal{P} \aleph_\mu = -m \mathcal{P} \partial_\mu \Upsilon. \quad (29)$$

While the first part (19) of the Ansatz simplifies the spin connection \aleph_μ to the gradient of a scalar function, the second part (26) specifies the affine connection. The simplest form of Eq. (29) is $T^{00} \partial_i g_{00} = 2m \mathcal{P} \partial_i \Upsilon$, which immediately leads to

$$g_{00} = 1 + 2(m \mathcal{P} / T_{00}) \Upsilon \rightarrow 1 + 2\Upsilon_{grav}, \quad (30)$$

Thus, we have *derived* the key formulae (11) and (30), which usually *are postulated* as the starting point of general relativity. Therefore, all fundamental predictions of general relativity are preserved. The field \aleph_μ , is indistinguishable from the gravitational field and is locally equivalent to the field of inertial forces. Eq. (29) explicitly states that it is possible to choose such a space-time coordinates (metric) that a small body moves along the geodesic lines of *this* metric.

7. A simple model with the axial potential (25) and only one mass/energy parameter appears to be solvable exactly. Large and small components of the *radially polarized* and spherically symmetric modes of the Dirac field behave as $r \rightarrow 0$ like $F \sim \cos(Q/r)$ and $G \sim \sin(Q/r)$, so that the probability density $R = r^2 j^0 = F^2 + G^2$ remains nearly constant within the range of sharp localization. These normalizeable bound states have a continuous energy spectrum when $-\infty < EQ < -3/8$. For the Dirac field with this simplest geometry we have $d(F^2 + G^2)/dr = 4mFG = 2m\mathcal{P}$, which provides a simple formula for the gravitational mass in Eqs. (25), (29) and (30), i.e.,

$$2m \int \mathcal{P} dV = R(0),$$

and allows one to conclude that, at the microscopic level, the gravitational mass is proportional to the *peak amplitude of the localization* of the Dirac field at the gravitating centers. The smallest particles are the heaviest, and on a purely dimensional ground we have $R(0) \sim |E|$. This is evidence that, even being of a different physical origin, the inertial ($\sim \int T^{00} dV \sim E$) and the gravitational ($\sim m \int \mathcal{P} dV \sim R(0)$) masses of stable particles are the same up to a possible factor which can be absorbed into “Newton’s gravitational constant” $G_N \sim (g^2/M^2)$; this suffices to guarantee the universality of free fall. Only an extensive study of Eqs. (23) and (24) can clarify whether this factor is universal. It looks like the

mass, which corresponds to this gravitational constant, is smaller than the formal “dimensional” Planck mass, $M_{Planck}^2 = (\hbar c/G_N)$, by a factor of g related to the electro-weak interactions. We suggest that the electro-weak and gravitational constants uniquely determine the mass M of the axial field.

8. Equations (23), (24) and (11), (29) clearly support an image of particles as moving singularities which *are shared by the Dirac and gravitational fields*. These equations guarantee that the theory is protected from *mathematical singularities* which inevitably appear when the nonlinear Einstein’s equations (11) are solved independently [6]. A striking agreement between the character of the motion of singular domains of the Einstein and the Dirac fields seems to be an indication that general relativity naturally requires matter in the form of the Dirac field. No other fields can provide the degree of localization, which is necessary for such an agreement. The complementarity of these two fields indeed solves the problem of motion as it was posed by Einstein [6]. In agreement with Einstein’s eventual judgement, the field equation (11) has no energy-momentum tensor of matter in its r.h.s. *The physical origin of the macroscopic forces of gravity between any two bodies is a trend of the global Dirac field to concentrate around the microscopic domains where this field happens to be extremely localized.* These forces tend to polarize the matter at the level of its spinor organization and they well may contribute at various stages of matter evolution. The picture of gravity as the effect of the axial field explains gravity as a coherent effect that cannot be screened by any bodies or fields. The “falling onto a centre” is universal and is guaranteed by a special position of the singular Newton’s force as a potential in the Dirac equation. It would be fair to say that the Dirac field is such a natural “stuffing” for Einstein’s singularities, that the prevailing feeling of the incompleteness of the Einstein-Infeld theory fades away. Of all possible solutions of the Einstein’s equations, only those that have the material partners among the compact solutions of the Dirac equation with the axial field in spin connection, are physically meaningful.

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